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AUTHOR(S):

TAMURA, Jun-ichi

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# Hankel determinants and substitutions – some results and problems

Jun-ichi TAMURA

3-3-7-307 Azamino Aoba-ku Yokohama 225-0011 Japan

1. Introduction. Let  $A^*$  be a free monoid generated by a non-empty set  $A$ , i.e.,  $A^*$  is the set of finite words over  $A$  with the empty word  $\lambda$ . We put  $A^* := A^* \cup A^{\mathbb{N}}$ , where  $\mathbb{N}$  is the set of non-negative integers, so that  $A^{\mathbb{N}}$  is the set of infinite words over  $A$ . Any monoid morphism  $\sigma: A^* \rightarrow A^*$  can be extended to a map  $\sigma: A^* \rightarrow A^*$  by  $\sigma(a_0 a_1 a_2 \cdots) := \sigma(a_0) \sigma(a_1) \sigma(a_2) \cdots$  ( $a_n \in A$ ), which is a so-called substitution (over  $A$ ). We say that  $\sigma$  is of constant length  $k$  iff  $\sigma(x)$  is a finite word of length  $k$  for all  $x \in A$ . A fixed point of a substitution  $\sigma$  is an infinite word  $\varphi \in A^{\mathbb{N}}$  satisfying  $\sigma(\varphi) = \varphi$ .

The fixed point of a substitution  $\sigma$  over  $\{a, b\}$  defined by

$$\sigma(a) = ab, \sigma(b) = ba \text{ (resp., } \sigma(a) = ab, \sigma(b) = a)$$

prefixed by  $a$  is referred to as the Thue-Morse word (resp., the Fibonacci word).

Let  $q \geq 1$  be an integer. We denote by  $\text{ord}_q(n)$  the largest integer  $e \geq 0$  such that  $n$  is divisible by  $q^e$ . We say a word  $w = w_1 w_2 w_3 \dots$  is a  $q$ -adic Toeplitz word iff  $w_m = w_n$  holds for any positive integers  $m, n$  satisfying  $\text{ord}_q(m) = \text{ord}_q(n)$ . Let  $\sigma$  be a substitution over an infinite alphabet  $A_{\infty} := \{a_0, a_1, a_2, \dots\}$  defined by

$$\sigma(a_n) = a_0 a_{n+1} \quad (n = 0, 1, 2, \dots).$$

For some of the symbols  $a_0, a_1, a_2, \dots$ , we also write  $a_0 = a, a_1 = b, a_2 = c$ , etc.

The substitution  $\sigma$  has a unique fixed point

$$\omega = abacabadabacabaeabacabadabacabaf\dots,$$

which is a 2-adic Toeplitz word. Any 2-adic Toeplitz word over a finite or an infinite alphabet  $B$  can be written by

$$\tau(\omega) := \tau(\omega_1) \tau(\omega_2) \tau(\omega_3) \dots,$$

where  $\tau$  is a map from  $A_{\infty}$  to  $B$ , and  $\omega =: \omega_1 \omega_2 \omega_3 \dots$  ( $\omega_i \in A_{\infty}$ ). In this sense the word  $\omega$

is a universal 2-adic Toeplitz word.

Our objective is to get something interesting related to determinants

$$H_n^{(m)} = H_n^{(m)}[\varphi] := \det(\varphi_{m+i+j})_{0 \leq i \leq n-1, 0 \leq j \leq n-1},$$

$$H_n = H_n[\varphi] := H_n^{(0)}[\varphi]$$

for a given infinite word  $\varphi = \varphi_0 \varphi_1 \varphi_2 \dots$  ( $\varphi_i \in A$ ) over a finite, or an infinite alphabet  $A$ , where  $H_n^{(m)}$  is considered to be an element of  $\mathbb{Z}[A]$ , i.e., a polynomial in independent variables  $\in A$  with integer coefficients.  $H_n^{(m)}[\varphi]$  can be extended to  $(m, n) \in \mathbb{Z} \times \mathbb{N}$  by setting  $H_n^{(0)} := 1$ ,  $\varphi_m := 0$  ( $m < 0$ ), where  $\mathbb{N}$  denotes the set of non-negative integers. In the following two sections, we give a very rough survey on the results related to  $H_n^{(m)}$ . In Section 2, we give some results on  $H_n^{(m)}[\varphi]$  for a general word  $\varphi$ , cf. [K-T-W], [T2]. In Section 3, we give some of the results related to  $H_n^{(m)}[\varphi]$  when  $\varphi$  is the Thue-Morse word, the Fibonacci word, and a Fibonacci-type word, cf. [A-P-ZXW-ZYW], [K-T-W], [T2]. In Section 4, we give a new characterization of the 2-adic Toeplitz words  $\varphi$  by an algebraic property (completely reducibility) of  $H_n[\varphi]$ , cf. [M-T-Tn]. We shall give no proofs, but state only results with minimum definition.

2. General properties of  $H_n[\varphi]$ . The set  $A^*$  becomes a complete metric space with respect to the metric defined by

$$d(\xi, \eta) := \exp(-\inf\{n; \xi_n \neq \eta_n\}) \quad (\xi = \xi_0 \xi_1 \xi_2 \dots, \eta = \eta_0 \eta_1 \eta_2 \dots \in A^* \quad (\xi_n, \eta_n \in A)).$$

As usual,  $K((Z))$  denotes the set of formal Laurent series of one variable  $Z$  over a field  $K$ . We put

$$K := \mathbb{Q}(A) \quad (\supset A).$$

The set  $K((Z^{-1}))$  becomes a metric space induced by a non-Archimedean norm defined by

$$\|\varphi^{(h)}\| := \exp(-n_0 + h), \quad n_0 := \inf\{n \in \mathbb{N}; \varphi_n \neq 0\} \quad (\|0\| := 0)$$

for

$$\varphi^{(h)} = \sum_{n \geq 0} \varphi_n Z^{-n+h} \in K((Z^{-1})) \quad (1)$$

with  $h \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ . Note that  $\|\varphi^{(h)}\| = \exp h$  holds if  $\varphi = \varphi_0 \varphi_1 \varphi_2 \dots \in A^{\mathbb{N}} \subset$

$K^*$ . If  $\varphi$  is a finite word of length  $k$ , then  $\varphi_n := 0$  for  $n \geq k$ .) For any given  $\varphi = \varphi_0 \varphi_1 \varphi_2 \dots \in K^*$ , we say that  $(P, Q) \in K[z]^2$  is an  $h$ -Padé pair of order  $m$  for  $\varphi$  iff

$$\|Q\varphi^{(h)} - P\| < \exp(-m), \quad Q \neq 0, \quad \deg Q := \deg_z Q \leq m \quad (2)$$

holds. The usual Padé pair (for a formal Laurent series) agrees with the  $h$ -Padé pair with  $h = -1$  (for a word), cf. [N-S]. It is known that an  $h$ -Padé pair  $(P, Q)$  of order  $m$  for  $\varphi$  always exists for any  $h \in \mathbb{Z}$ ,  $m \geq 0$ ,  $\varphi \in K^*$ , cf. Lemma 1, [T2]. For  $h$ -Padé pairs  $(P, Q)$  of order  $m$  for  $\varphi$ , a rational function  $P/Q \in K(z)$  is uniquely determined for any given  $h \in \mathbb{Z}$ ,  $m \geq 0$ ,  $\varphi \in K^*$ . The element  $P/Q \in K(z)$  for an  $h$ -Padé pair  $(P, Q)$  of order  $m$  for  $\varphi \in K^*$  is referred to as the  $h$ -Padé approximant of order  $m$  for  $\varphi$ . A number  $m \in \mathbb{N}$  is called a normal  $h$ -index for  $\varphi \in K^*$  if (2) implies  $\deg Q = m$ . A normal  $h$ -Padé pair, i.e.,  $\deg Q$  is a normal  $h$ -index, is said to be normalized if the leading coefficient of  $Q$  equals one. Normal  $(-1)$ -indices (resp.  $(-1)$ -Padé pairs,  $(-1)$ -Padé approximants) will be simply referred to as normal indices (resp. Padé pairs, Padé approximants). The set of all the normal  $h$ -indices for  $\varphi$  will be denoted by

$$\Lambda_h(\varphi) := \{m \in \mathbb{N}; m \text{ is normal } h\text{-indices for } \varphi\},$$

$$\Lambda(\varphi) := \Lambda_{-1}(\varphi).$$

We can consider the series (1) over  $K = \mathbb{Q}(a, b, \dots)$  with  $a, b, \dots \in \mathbb{C}$ . In such a case,  $\varphi^{(h)}$  defined by (1) turns out to be not only an element of  $\mathbb{C}((z^{-1}))$ , but also an analytic function on  $\{z \in \mathbb{C}; |z| > 1\}$ , and the  $h$ -Padé approximant of order  $m$  for  $\varphi^{(h)}$  pointwise converges to  $\varphi^{(h)}$  with respect to the usual topology on  $\mathbb{C}$  for each  $z \in \mathbb{C}$ ,  $|z| > 1$  as  $m$  tends to infinity.

**Proposition 1** (cf. [T1]). Let  $\varphi \in K^*$  be a word over  $K = \mathbb{Q}(A)$  with an alphabet  $A$  possibly consisting of infinite letters.

$$H_{h+1}^{(m)}[\varphi] = (-1)^{\lfloor m/2 \rfloor} \prod_{i=0}^{\lfloor m/2 \rfloor} P(z) \quad (h, m \in \mathbb{Z}, m \geq 0),$$

where  $(P, Q)$  is a normalized  $h$ -Padé pair of degree  $m$  for  $\varphi$ ,  $\lfloor x \rfloor$  denotes the largest integer not exceeding a real number  $x$ , and  $\prod_{i=0}^{\lfloor m/2 \rfloor}$  indicates a product

taken over all the zeros  $z$  of  $Q$  with their multiplicity in any field  $\widetilde{K}$  containing an algebraic closure of  $K$ .

Remark 1. We can take  $\widetilde{K} = \mathbb{C}$  in Proposition 1 in the case where  $A$  is a subset (possibly empty) of  $\mathbb{C}$ .

Remark 2. If  $m$  is not a normal  $h$ -index of  $\varphi$ , then  $P, Q \in K[z]$  have common zeros. Hence, it follows from Proposition 1 that  $m \notin \Lambda_n(\varphi)$  implies  $H_{n+1}^{(m)} = 0$ . The converse of this fact is valid, cf. Lemma 2, [T2].

In particular, Proposition 1 holds for all the fixed point  $\varphi \in A^{\mathbb{N}}(CK^*)$  of a substitution over any alphabet  $A$ . The following remark is useful, while it is valid only for a word  $\varphi$  consisting of at most two symbols.

Remark 3. Let  $M$  be a matrix of size  $n \times n$  with entries consisting of two variables  $a, b$  (symbols). Then

$$\det M = (a-b)^{n-1}(pa+qb) \in \mathbb{Z}[a, b],$$

where  $p, q$  are integers defined by

$$p = \det M \mid (a, b) = (1, 0), \quad q = \det M \mid (a, b) = (0, 1).$$

### 3. Thue-Morse, and Fibonacci cases.

J.-P. Allouche, J. Peyrière, Z.-X. Wen and Z.-Y. Wen considered  $H_{n,m}(\zeta)$  for the Thue-Morse sequence  $\zeta = \text{abbabaab} \dots$  with  $(a, b) = (1, 0)$ , and showed that the 2-dimensional word  $H_n^{(m)}(\zeta) \pmod{2}$  of  $(n, m) \in \mathbb{N}^2$  is 2-dimensionally automatic; it is remarkable that  $\Lambda(\zeta) = \mathbb{N}$  is known, cf. [A-P-ZXW-ZYW].

In general, it is very difficult to give an explicit formula of  $H_n^{(m)}(\varphi)$  for a given infinite word  $\varphi$  that is not periodic, while explicit formulae of  $H_n^{(m)}(\eta)$  are completely given for the Fibonacci word  $\eta = \text{abaabab} \dots$ , cf. Theorems 1-5 in [K-T-W]. By  $f_n$  we denote the  $n$ -th Fibonacci number ( $f_{-1} = f_0 = 1$ ,  $f_n = f_{n-1} + f_{n-2}$ ). Let

$$n = \sum_{i \geq 0} \delta_i(n) f_i \quad (\delta_i(n) \in \{0,1\}, \delta_{i+1}(n)\delta_i(n)=0 \text{ for all } i \geq 0)$$

be the representation of  $n$  in the Fibonacci base due to Zeckendorf. We write

$$m \equiv_k n$$

iff  $\delta_i(m) = \delta_i(n)$  holds for all  $0 \leq i < k$ . We put

$$\chi(k, S) := -1 \text{ if } k \equiv s \pmod{6} \text{ for an } s \in S,$$

$$1 \text{ otherwise,}$$

for a subset  $S$  of  $\{0,1,2,3,4,5\}$ . Then we have, for instance,

Proposition 2 (Theorem 3, [K-T-W]). For any  $k, m, i \geq 0$  integers satisfying

$$m \equiv_{k+1} i, \quad 0 \leq i \leq f_{k+1}-1,$$

the following formulae hold:

$$\begin{aligned} H_{f_k}^{(m)}[\eta] \mid (a, b) = (1, 0) &= \chi(k; 2) \chi(k; 1, 4)^i f_{k-1}, \\ &\text{if either } \delta_{k+1}(m) = 0 \text{ and } 0 \leq i < f_{k-1}, \\ &\text{or } \delta_{k+1}(m) = 1 \text{ and } 0 \leq i < f_k, \\ &= \chi(k; 1, 2, 4) f_{k-2}, \\ &\text{if either } \delta_{k+1}(m) = 0 \text{ and } i = f_{k-1}, \\ &\text{or } i = f_{k+1} - 1, \\ &= 0 \text{ otherwise,} \\ H_{f_k}^{(m)}[\eta] \mid (a, b) = (0, 1) &= \chi(k; 1, 2, 4) \chi(k; 1, 4)^i f_{k-2}, \\ &\text{if either } \delta_{k+1}(m) = 0 \text{ and } 0 \leq i < f_{k-1}, \\ &\text{or } \delta_{k+1}(m) = 1 \text{ and } 0 \leq i < f_k, \\ &= \chi(k; 2) f_{k-3}, \\ &\text{if either } \delta_{k+1}(m) = 0 \text{ and } i = f_{k-1}, \\ &\text{or } i = f_{k+1} - 1, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Notice that Proposition 2 together with Remark 3 gives a part of the explicit formulae for  $H_n^{(m)}[\eta]$ . In comparison with the automacity result for the 2-dimensional word  $(H_n^{(m)}(\zeta) \pmod{2})_{(n,m) \in \mathbb{N}^2}$  for the Thue-Morse sequence

$\zeta=10010110\cdots$  given in [A-P-ZXW-ZYW], we gave an explicit expression of the 2-dimensional word  $H_n^{(m)}(\eta)$   $(n,m)\in\mathbb{N}^2$  for the Fibonacci word  $\eta=abaabab\cdots$ , which is rather complicated, cf. Theorem 5 in [K-T-W].

In [T2], we developed a theory of analysis on words, especially for words of Fibonacci type, i.e. the fixed points of substitutions of the form

$$\sigma(a):=a^k b, \sigma(b):=a \quad (k>0). \quad (3)$$

We denote by  $\varepsilon=\varepsilon(a,b;k)$  the fixed point of the substitution defined by (3), and by  $|w|_x$ :=the number of occurrences of an identical symbol  $x$  appearing in a word  $w$ . In Sections 2, 3 in [T2], we gave explicit formulae for the continued fraction expansion with partial denominators  $\in K[z]$  ( $K=\mathbb{Q}(a,b)$ ) and normalized Padé pairs for the Laurent series  $\varepsilon^{(-1)}(z)=\varepsilon(z;a,b;k)$  defined by (1) with  $\varphi=\varepsilon$ , cf. Theorems 1-12, [T2]. For instance, we have

Proposition 3 (Theorem 8, [T2]). Let  $k\geq 2$ . The continued fraction expansion of the Laurent series  $\varepsilon^{(-1)}(z)\in K((z^{-1}))$  for  $\varepsilon=\varepsilon(a,b;k)\in\{a,b\}^\mathbb{N}$  is given by

$$\varepsilon^{(-1)}(z)=[0;a^{-1}(z-1), (-1)^m(a-b)^{-1}h_m^2b_{m-1}^*, (-1)^m(a-b)h_m^{-1}h_{m+1}^{-1}(z-1)]_{m=0}^\infty,$$

where

$$\begin{aligned} h_n &:= |\sigma^n(a)|_a a + |\sigma^n(a)|_b b (=g_n a + g_{n-1} b \in \mathbb{Z}[a,b]), \quad f_n = f_{n,k} := |\sigma^n(a)|, \\ b_n^* &:= b_n^*(z;k) := z^{f_n} \sum_{0 \leq i \leq f_{n+1}-1} z^i \cdot \sum_{1 \leq j \leq k-1} (k-j) z^{(j-1)f_{n+1}+k} \sum_{0 \leq i \leq f_n-1} z^i \in \mathbb{Z}[z]. \end{aligned}$$

If  $(a,b)\in\mathbb{C}^2$ , then Proposition 3 is valid under the condition

$$a \neq b, \quad h_n (=g_n a + g_{n-1} b) \neq 0 \quad \text{for all } n \geq 0. \quad (4)$$

We can give explicit formulae for the continued fraction expansion for  $\varepsilon^{(-1)}(z)=\varepsilon(z;a,b;k)\in\mathbb{C}((z^{-1}))$  with  $(a,b)\in\mathbb{C}^2$ , which does not satisfy (4). For example,

Proposition 4 (Theorem 9, [T2]). Let  $(a,b)\in\mathbb{C}^2$  with  $h_1=0$ ,  $a\neq 0$ ,  $t\geq 0$ . Then

$$\begin{aligned} \varepsilon^{(-1)}(z) &= \varepsilon^{(-1)}(z;a,-g_{t-1}^{-1}g_t a) \\ &= [0;a_1,d_{-1},c_0,d_0,\dots,c_{t-2},d_{t-2},e_1,e_2,i_m,j_m]_{m=0}^\infty \end{aligned}$$

holds with partial denominators  $\in \mathbb{C}[z]$  given by

$$\begin{aligned} a_1 &= a^{-1}(z-1), \\ c_m &= (-1)^m(a-b)h_m^{-1}h_{m+1}^{-1}(z-1), \end{aligned}$$

$$d_m = (-1)^{m+1} (a-b)^{-1} h_{m+1}^2 b_m^*,$$

$$e_1 = (-1)^{t-1} (a-b) h_{t-1}^{-2} (z-1) b_{t-1},$$

$$e_2 = (-1)^{t-1} (a-b)^{-1} h_{t-1}^2 b_t^*,$$

$$i_m = (-1)^{m+t-1} (a-b) h_{t-1}^{-2} g_m^{-1} g_{m+1}^{-1} (z-1),$$

$$j_m = (-1)^{m+t} (a-b)^{-1} h_{t-1}^2 g_{m+1}^2 b_{m+t+1}^*.$$

Concerning such continued fractions, we studied uniform convergence in Section 5, [T2]. Related to the product formula (Proposition 1), we studied the distribution, and the simplicity of the zero points of  $Q(z)$  for the Padé pairs  $(P, Q)$  for  $\varepsilon(z; a, b; k)$  in Section 4, [T2].

It is an interesting problem which asks whether we can do the same for the Thue-Morse word  $\xi$ . The fact  $\Lambda(\xi) = N$  (cf. [A-P-ZXW-ZYW]), we mentioned, says that all the denominators of the continued fraction for  $\xi^{(-1)}(z)$  are of degree 1. It is of special interest to find the the continued fraction expansion for  $\xi^{(-1)}(z)$  in a closed form.

We could not give a completely explicit formula for  $H_n^{(m)}[\varepsilon(z; a, b; k)]$  when  $k \geq 2$ ; we gave the following

Proposition 5 (cf. Corollary 6, [T2]).

$$H_{f_n}^{(0)}[\varepsilon(z; a, b; k)] = r_n (g_n a + g_{n-1} b) (a-b)^{f_n-1},$$

$$H_{f_{n+1}-1}^{(0)}[\varepsilon(z; a, b; k)] = s_n (g_n a + g_{n-1} b) (a-b)^{f_{n+1}-2} \quad (n \geq 0);$$

and  $H_m^{(0)} = 0$  for all  $m \neq f_n$  and  $m \neq f_{n+1}-1$  ( $n \geq 0$ ), where  $r_n \neq 0$ ,  $s_n \neq 0$  are integers independent of  $a, b$ .

4. Toeplitz cases. In this section, we consider Hankel determinants for 2-adic Toeplitz words

$$w = w_1 w_2 w_3 \dots \quad (w_i \in A).$$

Note that the numbering of the symbols starts from 1 (not from 0), cf. Sections 2, 3. Recall that a word  $w = w_1 w_2 w_3 \dots$  ( $w_i \in A$ ) is a 2-adic Toeplitz word iff



$$\text{ord}_2(m) = \text{ord}_2(n) \implies w_m = w_n.$$

Without loss of generality, we may suppose  $A = \{a_0, a_1, a_2, \dots\}$ . In some cases, we use symbols  $a, b, c, \dots$  instead of  $a_0, a_1, a_2, \dots$  as before. Recall also the universal 2-adic Toeplitz word

$$\omega = \text{abacabadabacabaeabacabadabacabaf} \dots$$

and the notation

$$H_n[\omega] := H_n^{(0)}[\omega]$$

defined in Section 1. For example, by direct calculation, we have

$$\begin{aligned} H_7[\omega] &= \begin{vmatrix} a & b & a & c & a & b & a \\ b & a & c & a & b & a & d \\ a & c & a & b & a & d & a \\ c & a & b & a & d & a & b \\ a & b & a & d & a & b & a \\ b & a & d & a & b & a & c \\ a & d & a & b & a & c & a \end{vmatrix} \\ &= -4ab^2c^4 + abc^5 - ac^6 + 16ab^2c^3d - 12abc^4d + 2ac^5d - 24ab^2c^2d^2 \\ &\quad + 8abc^3d^2 + ac^4d^2 + 16ab^2cd^3 + 8abc^2d^3 - 4ac^3d^3 - 4ab^2d^4 - 12abcd^4 \\ &\quad + ac^2d^4 + 4abd^5 + 2acd^5 - ad^6 \\ &= -a(2b-c-d)^2(c-d)^4. \end{aligned}$$

We say that a form (i.e., a homogeneous polynomial)  $P \in \mathbb{Z}[A]$  is completely reducible iff  $P=0$ , or  $P$  can be factorized into linear forms  $\in \mathbb{Z}[A]$ , i.e.,

$$P = P_1 P_2 \cdots P_k \quad (\deg P_i = 1 \text{ for all } 1 \leq i \leq k).$$

One can check that  $H_n[\omega]$  are non-zero completely reducible forms for small  $n$  (for  $n \leq 30$  or so) by using the soft "Mathematica". This is a curious phenomenon, since, for instance,  $H_2[w]$  (resp.,  $H_3[w]$ , etc.) is not completely reducible for any word  $w$  having  $abc$  (resp.,  $abacd$ , etc.) as its prefix of  $w$ . Related to such a phenomenon, we can show the following

Proposition 6 (Main Theorem in [M-T-Tn]). Let  $w$  be a fixed point of a substitution of constant length 2. Suppose  $w$  is a word strictly over an alphabet

consisting of at least 3 symbols. Then  $H_n[w]$  is completely reducible for all  $n \geq 1$  if and only if  $w$  is a 2-adic Toeplitz word.

The proof of this proposition together with something more interesting (probably) will appear in the forthcoming paper [M-T-Tn].

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